Q) Solve the NLPP by Lagrangian Multiplier method.

$\min Z = 6x_1^2 + 5x_2^2$

s.t.

$x_1 + 5x_2 = 3$

and $x_1, x_2 \geq 0$

Solution:

Here, $f(X) = 6x_1^2 + 5x_2^2$

$g(X) = x_1 + 5x_2 = 3$

$\Rightarrow h(X) = g(X) - 3$

$\therefore h(X) = x_1 + 5x_2 - 3$

Let λ be a constant called Lagrangian Multiplier.

Then the Lagrangian funⁿ:

$$L(X, \lambda) = f(X) - \lambda h(X)$$
$$= 6x_1^2 + 5x_2^2 - \lambda(x_1 + 5x_2 - 3)$$

$\therefore$ for existence of maxima and minima of $f(X)$, we have

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 12x_1 - \lambda = 0 \quad \text{---} (1)$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 10x_2 - 5\lambda = 0 \quad \text{---} (2)$$

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow x_1 + 5x_2 - 3 = 0 \quad \text{---} (3)$$

From ①,

$$12x_1 = \lambda$$

② $\Rightarrow$ $10x_2 - 5(12x_1) = 0$

$10x_2 - 60x_1 = 0$

$6x_1 - x_2 = 0$

$6x_1 - x_2 = 0$

$x_1 + 5x_2 = 3$ (from ③)

$\cancel{6}x_1 - x_2 = 0$

$6x_1 + 30x_2 = 18$

$-31x_2 = -18$

$$\boxed{x_2 = \frac{18}{31}}$$

$x_1 = \frac{x_2}{6} \Rightarrow x_1 = \frac{18}{31} \times \frac{1}{6} \Rightarrow \boxed{x_1 = \frac{3}{31}}$

$\lambda = 12x_1$

$= 12 \times \frac{3}{31}$

$$\boxed{\lambda = \frac{36}{31}}$$

$\therefore$ the Stationary point $(x_0, \lambda_0) = (x_1, x_2, \lambda)$

$$= \left(\frac{3}{31}, \frac{18}{31}, \frac{36}{31}\right)$$

Now the Bordered Hessian Matrix is

$$H^B = \left[\begin{array}{c|cc} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} \\ \hline \frac{\partial h}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_2^2} \end{array}\right]$$

$$= \begin{bmatrix} 0 & 1 & 5 \\ 1 & 12 & 0 \\ 5 & 0 & 10 \end{bmatrix}$$

$\therefore$ The value of the Hessian matrix at the stationary point

$$H_0^B = \begin{bmatrix} 0 & 1 & 5 \\ 1 & 12 & 0 \\ 5 & 0 & 10 \end{bmatrix}$$

Also, $n = 2 \rightarrow$ no. of variables

$m = 1 \rightarrow$ no. of constraints

and. $n - m = 2 - 1 = 1$

$2m + 1 = 3$

$\Big\}$ Refer page 8 Point 2

$\therefore$ The last Principal minor of order 3.

ie. $\Delta_3 = \begin{vmatrix} 0 & 1 & 5 \\ 1 & 12 & 0 \\ 5 & 6 & 10 \end{vmatrix}$

$= -1(10-0)+5(0-60)$

$= -10-300$

$= -310 < 0$

Now.

$(-1)^m = (-1)^1 = -1 < 0$

$\Rightarrow$ The last principal minor and $(-1)^m$ are of the same sign.

Therefore, the value of the function $f(X)$ is minimum at the stationary point $\left(\frac{3}{31}, \frac{18}{31}, \frac{36}{31}\right)$ and the minimum value is

$f(X) = 6x_1^2 + 5x_2^2$

$= 6 \times \left(\frac{3}{31}\right)^2 + 5 \times \left(\frac{18}{31}\right)^2$

$= \frac{54}{31}$